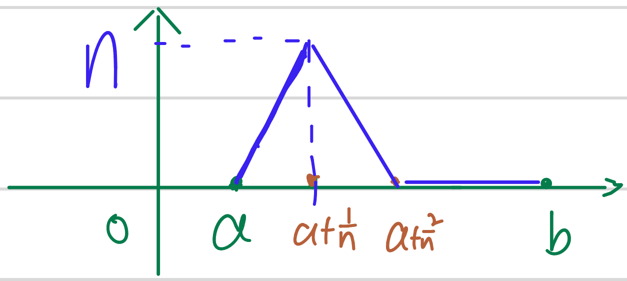
Solution 2

1. (i) Let

$$f_n(x) = \begin{cases} n^2(x-a), & a \le x \le a+\frac{1}{n} \\ -n^2x + an^2 + 2n, & a+\frac{1}{n} < x < a+\frac{2}{n} \\ 0 & a+\frac{2}{n} \le x \le b, \end{cases} \quad \forall n \in N$$



$$\|f_n\|_1 = \int_a^b |f_n(t)|\,dt = n \cdot \frac{2}{n} \cdot \frac{1}{2} = 1, \quad \forall n \in N,$$

$$\|f_n\|_\infty = n$$

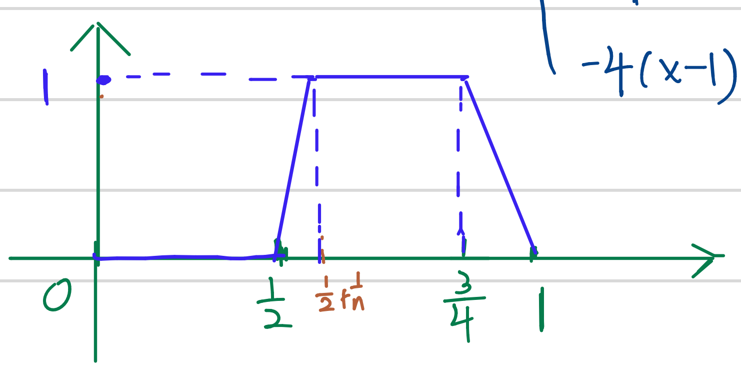
Thus, we can't find a constant $c > 0$, such that

$$C^{-1}\|f\|_1 \le \|f\|_\infty \le C\|f\|_1, \quad \forall f \in X.$$

$\|\cdot\|_\infty$ and $\|\cdot\|_1$ are not equivalent.

(ii) X is not Banach space. WLOG, we assume $[a,b]=[0,1]$

$$f_n(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2} \\ n(x-\frac{1}{2}), & \frac{1}{2} < x \le \frac{1}{2}+\frac{1}{n} \\ 1, & \frac{1}{2}+\frac{1}{n} < x \le \frac{3}{4} \\ -4(x-1), & \frac{3}{4} < x \le 1 \end{cases}, \quad n \ge 10, \ n \in N.$$



$\frac{1-0}{\frac{1}{4}}$

$$\|f_n - f_m\|_1 = \frac{1}{2} \cdot \left|\frac{1}{n} - \frac{1}{m}\right| \to 0, \quad \text{as } n, m \to \infty.$$

$\{f_n\}$ is a Cauchy sequence in $(X, \|\cdot\|_1)$

Let $f(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2} \\ 1, & \frac{1}{2} < x \le \frac{3}{4} \\ -4(x-1), & \frac{3}{4} < x \le 1 \end{cases}$

Then $\|f_n - f\|_1 = \frac{1}{2n} \to 0$ as $n \to \infty$.

However, $f \notin X$ and thus $(X, \|\cdot\|_1)$ is not a Banach space.

(iii) Let $f \in X$,

$$\|Tf\|_\infty = \max_{x \in [a,b]} |Tf(x)| = \max_{x \in [a,b]} \left| \int_a^x f(t)\,dt \right| \le \int_a^b |f(t)|\,dt = \|f\|_1$$

Thus, $\|T\| \le 1$.

On the other hand, Take $f_0 \equiv 1$ on $[a,b]$,

$$\|f_0\|_1 = \int_a^b 1\,dt = b-a$$

$$\|Tf_0\|_\infty = \max_{x \in [a,b]} \int_a^x f_0(t)\,dt = b-a = \|f\|_1$$

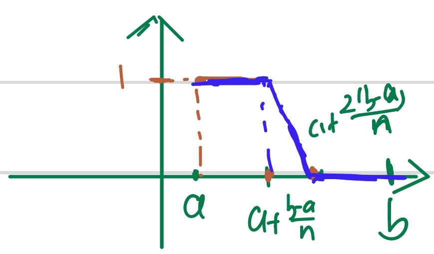
$\Rightarrow \quad \|T\| \ge 1$.

Therefore, $\|T\| = 1$.

(iv) $$\|Tf\|_1 = \int_a^b \left| \int_a^x f(t)\,dt \right| dx$$
$$\le \int_a^b \int_a^x |f(t)|\,dt\,dx$$
$$= \int_a^b \int_t^b |f(t)|\,dx\,dt$$
$$\le (b-a) \int_a^b |f(t)|\,dt = (b-a)\|f\|_1$$

Thus, $\|T\| \le b-a$

Let
$$f_n(x) = \begin{cases} 1, & a < x \le a + \frac{b-a}{n} \\ \frac{n}{b-a}(x-a) - 2, & a + \frac{b-a}{n} < x \le a + \frac{2(b-a)}{n} \\ 0, & a + \frac{2(b-a)}{n} < x \le b \end{cases}$$



$$\|f_n\|_1 = \frac{b-a}{n} + \frac{b-a}{2n} = \frac{3(b-a)}{2n}$$

$$Tf_n(x) = \begin{cases} x-a, & 0 < x \le a+\frac{b-a}{n} \\ \frac{3(b-a)}{4n} - \frac{n}{2(b-a)}[x-(a+\frac{b-a}{n})]^2, & a+\frac{b-a}{n} < x \le a+\frac{2(b-a)}{n} \\ \frac{3(b-a)}{2n} & a+\frac{2(b-a)}{n} < x \le b \end{cases}$$

$$\|Tf_n\|_1 = \int_a^b \left| \int_a^x f_n(t)\,dt \right| dx$$

$$= \int_a^{a+\frac{b-a}{n}} \left| \int_a^x f_n(t)\,dt \right| dx + \int_{a+\frac{b-a}{n}}^{a+\frac{2(b-a)}{n}} \left| \int_a^x Tf_n(t)\,dt \right| dx$$

$$+ \int_{a+\frac{2(b-a)}{n}}^{b} \left| \int_a^x Tf_n(t)\,dt \right| dx$$

$$= \frac{1}{6}\frac{(b-a)^3}{n^3} + \frac{(b-a)^2}{6n^2} + \frac{3(b-a)^2}{2n} - \frac{6(b-a)^2}{2n^2}$$

$$= \frac{3(b-a)}{2n} \cdot (b-a) \left[1 + \frac{b-a}{9n^2} + \frac{1}{9n} - \frac{2}{n} \right]$$

That is,

$$\|Tf_n\|_1 \ge \|f_n\|_1 \cdot (b-a) \left(1 + \frac{b-a}{9n^2} + \frac{1}{9n} - \frac{2}{n} \right)$$

Let $n \to \infty$, we have

$$\|T\| \ge b-a.$$

Therefore, $\|T\| = b-a$.

2. (i) Let $x_1, x_2 \in E$ such that

$T(x_1) = T(x_2)$ i.e. $x_1 + F = x_2 + F$,

Then $x_1 = x_2 + y$ for some $y \in F$.

which implies $x_1 - x_2 \in F$

Since $x_1, x_2 \in E$ and $E \cap F = \{0\}$

We have $y = 0$.

Therefore, T is injective.

For any $\bar{z} \in X/F$, say $\bar{z} = z + F$ for some $z \in X$.

- Claim: $\exists\, x \in E$, such that $T(x) = \bar{z}$. This implies T is surjective.

Since $X = E + F$, we may write z in the form $z = x + y$, with $x \in E$, $y \in F$.

Then $z + F = x + y + F = x + F$.

Thus, $T(x) = z + F = \bar{z}$.

- The linear property of map T is immediate from the definition of the linear operations in X/F, indeed, we have

$$(\alpha_1 x_1 + \alpha_2 x_2) + F = \alpha_1 (x_1 + F) + \alpha_2 (x_2 + F)$$

That is $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$

Claim: T is bounded.

For any $x \in E$,

$$\|Tx\|_{X/F} = \|\bar{x}\|_{X/F} = \inf\{\|x + v\|, : v \in F\}$$
$$= \inf\{\|x\| + \|v\| : v \in F\}$$
$$= \|x\|$$

Thus, $\|T\| = 1$.

Therefore, T is bounded linear isomorphism.

Let $\bar{z} \in X/F$, say $\bar{z} = z + F$ for some $z \in X$.

Write $z = x + y$ with $x \in E$ and $y \in F$.

$$\|\bar{z}\|_{X/F} = \inf\{\|z + v\| : v \in F\} = \inf\{\|x + y + v\| : v \in F\}$$
$$= \inf\{\|x + v\| : v \in F\} = \inf\{\|x\| + \|v\| : v \in F\}$$
$$= \|x\|$$

Then $\| T^{-1}(\bar{z}) \|_1 = \| x \|_1 = \| \bar{z} \|_{X/F}$,

$\Rightarrow \quad \| T^{-1} \| = 1$.

Therefore, T^{-1} is bounded.